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NECESSARY AND SUFFICIENT FRACTURE CRITERIA FOR A COMPOSITE WITH A BRITTLE MATRIX. PART 2. HIGH-STRENGTH REINFORCEMENT

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Experimental results on fracture of composite specimens with a brittle matrix are given. A modification is proposed for the shear model that describes deformation of a composite with failed matrix. A three-parameter sufficient criterion of quasibrittle strength of the composite is constructed, based on the equations of an orthotropic medium. Simple analytical expressions that relate the macrocrack length to the loading, structural, stiffness, and strength parameters of the composite medium are obtained. An approach for constructing multi-parameter criteria that take into account special features of deformation of the materials of the composite components is discussed.

Key words: *fracture criteria, composite, reinforcement, pre-fracture zone, multi-parameter criteria.*

Introduction. In the first part of this work [1], a pre-fracture zone of length Δ in weakly reinforced composites is studied on the basis of the Neuber–Novozhilov approach [2, 3]. The total stress-intensity factor (SIF) $K_{\rm I}^0$ in the Leonov–Panasyuk–Dugdale model [4, 5] can be equal to zero or positive: $K_{\rm I}^0 = 0$ or $K_{\rm I}^0 > 0$. The applicability regions of these restrictions are discussed in [6]. The fracture mechanism for a composite with a brittle matrix is as follows: a macrocrack grows due to the pre-fracture zone formed ahead of the initial-crack tip, which is followed by failure of the fiber nearest to the crack center [7]. Only after the fibers are broken is the composite separated into parts. To describe the fracture of composites reinforced with high-strength fibers, it is expedient to model the composite material by an orthotropic medium and formulate fracture criteria with allowance for the material structure [7, 8].

The pre-fracture zone adjacent to the crack tip is of primary interest. Below, the quantities that refer to the composite or matrix (before the matrix failure) and reinforcement fibers (after the failure) are denoted by the subscripts 1 and 2, respectively. Let $\sigma_{m2} = \text{const}$ be averaged stresses in the pre-fracture zone [1], which differ from the "theoretical" strength of a bundle of fibers with a brittle matrix σ_{m1}. The following three cases are possible: 1) $\sigma_{m1} \gg \sigma_{m2}$; 2) $\sigma_{m1} \simeq \sigma_{m2}$; 3) $\sigma_{m1} \ll \sigma_{m2}$. The first case corresponds to weak reinforcement [1] and the third case to high-strength fiber reinforcement; in all the cases, the condition $K_{\rm I}^0 = 0$ or $K_{\rm I}^0 > 0$ can be satisfied.

It appears natural to use the necessary criterion of brittle strength for the matrix and the sufficient criterion of quasibrittle strength for reinforcement (see [3]) in studying the processes of pre-fracture and ultimate fracture of a composite. The three-parameter criterion proposed in [1] is a strain-force criterion.

1. Physicomechanical Model of a Bundle of Fibers in the Pre-Fracture Zone for the High-Strength Fiber Reinforcement. Let the initial composite have a regular unidirectional structure [7–9] and r_1 be the distance between the fibers. To describe the composite material outside the pre-fracture zone, we use the equations of a homogeneous orthotropic medium [8]. In the pre-fracture zone, the behavior of the partly failed composite depends on its structure and σ–ε diagram of reinforcing fibers. The model of a bundle of fibers is the simplest model of the composite. Each representative volume of the composite contains a fiber and brittle matrix [7–9], the high-strength fibers can be brittle or plastic, and the limit elongation per unit length of the fiber ε_{m2}

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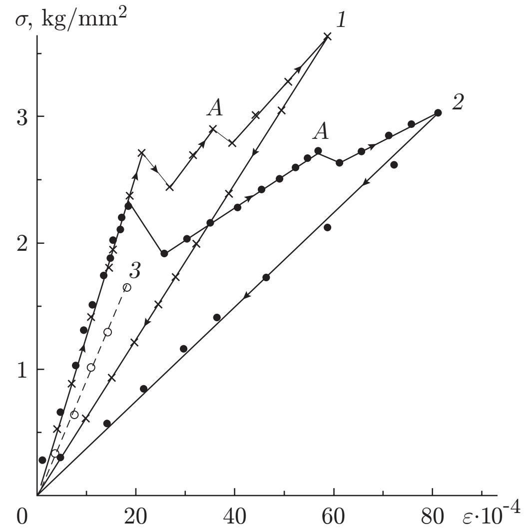

Fig. 1. Curves σ–ε for the specimens: 1) specimen with fibers from ÉF-32-301 glass-reinforced plastic ($\chi_2 = 0.125$); 2) specimen with fibers from ÉF-32-301 glass-reinforced plastic ($\chi_2 = 0.05$); 3) specimen from the matrix material.

is much higher than that of the matrix ε_{m1} ($\varepsilon_{m2} \gg \varepsilon_{m1}$). The simplified σ–ε diagrams of a bundle of fibers and their approximations corresponding to testing of a bundle of fibers of the composite in an extremely rigid machine are shown in [1, Fig. 1]. We consider the results of testing model composite specimens, which show that, in some cases, the simplified description [9] of force bridges by the shear model fails to describe specific features of composite deformation in the pre-fracture zone.

Figures 1 and 2 show the curves of extension and unloading of reinforced specimens from an epoxy matrix with a filler. The loading and unloading paths are shown by arrows and the jump due to crack formation is shown by dashed curves. To make the epoxy matrix brittle, sand was used as a filler. The specimens were blade-shaped and had a cross-sectional area at the measuring base $S = 60$–65 mm^2. The central part of the specimen was reinforced with: 1) rod made of ÉF-32-301 unidirectional glass-reinforced plastic; 2) steel wire; 3) copper wire. Extension of the specimens were performed in a lever-type machine. The strain was measured by clock-type indicators. The strength and stiffness characteristics of the matrix were determined using a reference specimen whose polymerization regime was the same as for the reinforced specimen. Figure 1 shows the deformation curves 1 and 2 for specimens with fibers from ÉF-32-301 glass-reinforced plastic with volume fractions of fibers $\chi_2 = 0.125$ and 0.050, respectively; curve 3 refers to deformation of the brittle matrix (χ_2 is the volume fraction of reinforcing fibers in the composite and $\chi_1 = 1 - \chi_2$ is the volume fraction of the matrix). The points refer to the experimental data. Figure 2 shows the deformation curves 1 and 2 for specimens reinforced with copper and steel wires, respectively (volume fractions of fibers $\chi_2 = 0.05$). Curves 3 and 4 refer to deformation of the brittle matrix of specimens reinforced with copper and steel wires. Curves 1 and 2 in Figs. 1 and 2 differ substantially since not only the brittle matrix failed but also a secondary failure of the bundle of fibers occurred at the point A on the deformation curve during the deformation of the specimens with fibers from ÉF-32-301 glass-reinforced plastic. Moreover, after unloading, crack closure occurred in specimens with fibers from glass-reinforced plastic, whereas a considerable gap remained between the crack edges in specimens reinforced with copper or steel wire owing to plastic properties of these materials.

Elastic modulus and strength of the tested specimens are determined with a reasonable accuracy by the mixture rule. The average stress σ in the unidirectional composite is determined by the relation [10]

$$\sigma = \sigma_1\chi_1 + \sigma_2\chi_2, \tag{1}$$

where σ_1 and σ_2 are the stresses in the matrix and fiber, respectively. For equal strains of the matrix and fiber (before the failure), the stresses are related as the elastic moduli:

$$\sigma_2/\sigma_1 = E_2/E_1 \tag{2}$$

(E_1 and E_2 are the elastic moduli of the matrix and fiber, respectively). We substitute the stiffness characteristics of a particular element and matrix into relations (1) and (2). Given their volume fractions, we find the fracture

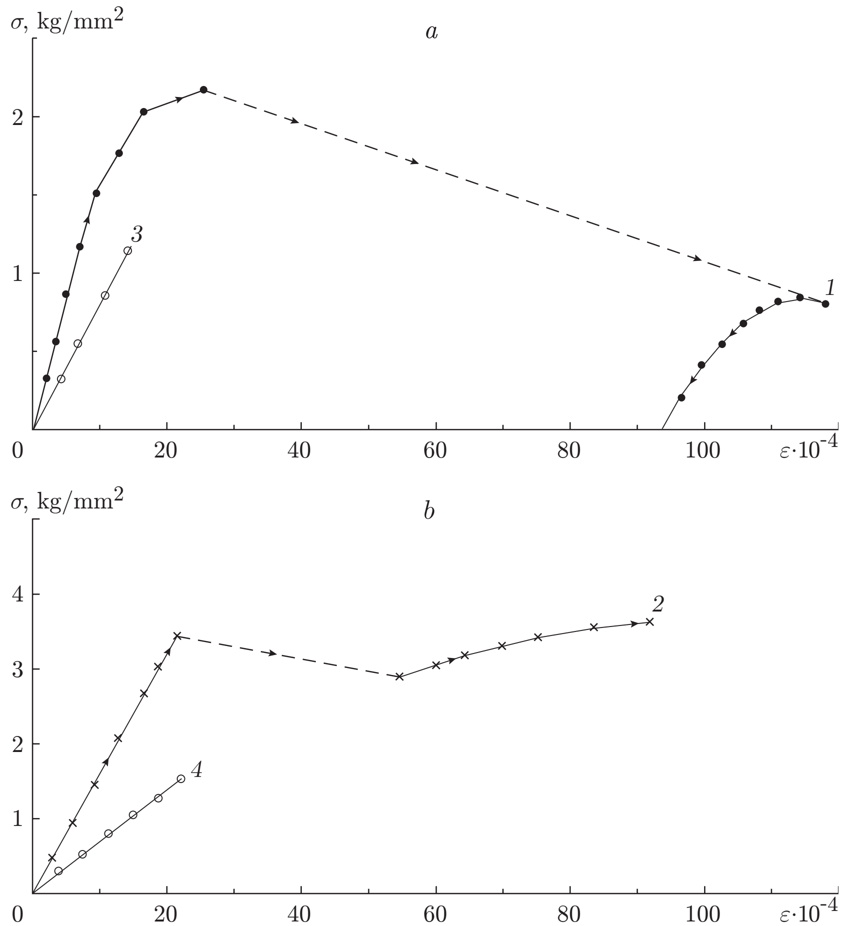

Fig. 2. Curves σ–ε for the specimens: (a) curve 1 refers to the specimen reinforced with a copper wire ($\chi_2 = 0.05$) and curve 3 refers to the specimen made of the matrix material; (b) curve 2 refers to the specimen reinforced with a steel wire ($\chi_2 = 0.05$) and curve 4 refers to the specimen made of the matrix material.

TABLE 1

Specimen No.	Matrix		Reinforcement		Composite			
					Experiment		Theory	
	E, kg/mm^2	σ_m, kg/mm^2	E, kg/mm^2	σ_m, kg/mm^2	E, kg/mm^2	σ_m, kg/mm^2	E, kg/mm^2	σ_m, kg/mm^2
1	800	1.73	4900	55	1060	2.67	1240	2.80
2	800	1.73	4900	55	1000	2.20	990	2.17
3	800	1.83	20,000	70	1670	3.53	1700	4.00
4	800	1.35	10,000	25	1300	2.25	1210	2.16

stress of the composite σ_{m1} by finding a relation between the quantities obtained and strength characteristics of the matrix. The failure of the composite is understood as the appearance of one or several cracks in the matrix of the specimen. In the cracked composite specimen reinforced with high-strength fibers, the fibers play the role of a force bridge between two parts of the composite specimen. The experimental results for specimens with only one crack are given below.

The fracture stresses calculated by the mixture rule are compared with experimental data in Table 1. The first and second specimens were reinforced with glass-reinforced plastic, and the third and fourth specimens were reinforced with steel and copper wires, respectively. The volume fraction of reinforcing fibers was $\chi_2 = 0.125$ for the first specimen and $\chi_2 = 0.05$ for the second, third, and fourth specimens. For the stresses that led to matrix

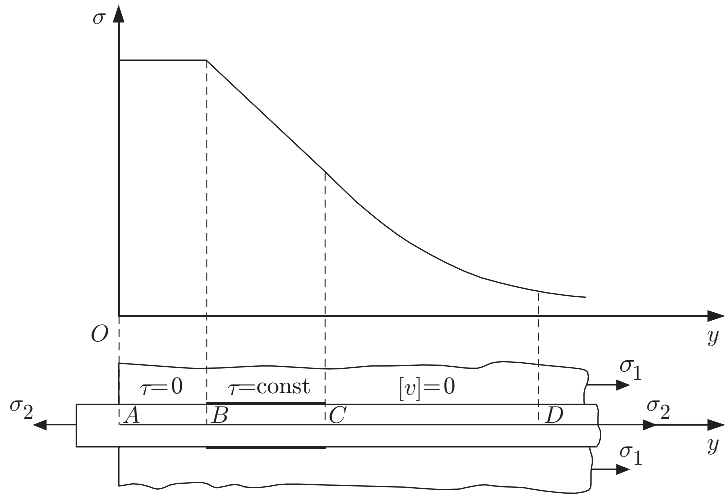

Fig. 3. Shear model of a composite with a site of reinforcement exfoliation.

cracking, the elongations per unit length of the composite were $\varepsilon_{m1} = 22 \cdot 10^{-4}$ for the first, second, and third specimens and $\varepsilon_{m1} = 25 \cdot 10^{-4}$ for the fourth specimen. The mixture rules predicts the appearance of cracks in the matrix with reasonable accuracy (see Table 1). After the crack was formed in the matrix, a jump from one branch of the deformation curve to another (descending portions of the curves in Figs. 1 and 2) was observed, since a relatively "soft" lever-type testing machine was used in the experiments. From the viewpoint of estimating the critical crack opening (CCO) of various specimens in the case where the force bridges fail, the σ–ε curves shown in Figs. 1 and 2 are of the most interest. The experimentally obtained CCO values are peculiar to composites with a brittle matrix. These values agree with the Budiansky–Evans–Hutchinson model [9] (see [7, 8]); however, the experimental CCO values exceed those obtained with the use of the shear model [9]. The cases where the σ–ε diagrams of the matrix and high-strength fiber materials have a considerable yield area are much more complex compared to the case where the fibers are characterized by a yield area and the matrix is brittle. Some theoretical generalizations used to formulate multi-parameter strength criteria are given below for the case where the σ–ε diagrams of the matrix and fiber materials have a pronounced yield area.

Let us determine the applicability range of the shear model [9]. The preliminary tests were performed for specimens with an epoxy matrix (in the solid state, epoxy is an optically active medium) and reinforcing rods made of a material characterized by clearly defined yield properties. In the neighborhood of the crack edges, the following three regions were determined: the region of the reinforcement-matrix exfoliation at a distance from the crack surface that exceeded significantly the diameter of the reinforcing rods [9]; the regions with partial slipping where dry friction occurs between the matrix and reinforcement; and the region of simultaneous deformation of the matrix and reinforcement.

It is expedient to consider the modified shear model [9] shown in Fig. 3 (cf. Fig. 3 of [9]). Figure 3 (below) shows the exfoliation regions AB (the shear stresses τ vanish at the matrix–reinforcement interface: $\tau = 0$), dry friction BC ($\tau = \text{const}$), and simultaneous deformation CD (there is no displacement jump $[v]$ at the matrix–reinforcement interface: $[v] = 0$). Figure 3 (above) shows the stress distribution along the reinforcement. The regions AB, BC, and CD refer to constant, linear, and exponentially decaying stresses in the reinforcement, respectively, and the point A lies at the crack surface. The authors consider that exfoliation can be caused by yielding of the rod material: in the region where the material yields, the rod diameter decreases, which leads to exfoliation. The CCO values obtained with the use of the initial and modified models differ considerably, since the modified model takes into account the parts of exfoliation and partial slippage.

We consider the cross-sectional size of the pre-fracture zone for unidirectional reinforcement. The reduced length of "free" reinforcement that sustains the force bonds between the crack flanks (Fig. 3) is denoted by a. Thus, the pre-fracture zone is a rectangle with sides Δ and a. We assume that a theoretical or experimental value of the CCO at the point $x = -\Delta$ is available (the coordinate origin is located at the right tip of the crack): $a_{m2} = (\varepsilon_{m2} - \varepsilon_{m1})a$. In the pre-fracture zone, constant stresses σ_{m2} act in the interval $[-\Delta, 0)$ [4, 5].

Thus, we have the geometric parameters r_1, Δ, and a_{m2} and force parameters σ_{m1} and σ_{m2} used to formulate the sufficient criterion of quasibrittle strength. This allows us to construct three-parameter fracture curves corresponding to the strain-force criterion. The geometric parameter r_1 characterizes the structure and is used to formulate the discrete-integral criterion, and the length of the pre-fracture zone Δ is determined from the restriction imposed on the CCO magnitude. Finally, we have one strain parameter a_{m2} and two force parameters σ_{m1} and σ_{m2} ($\sigma_{m1} \ll \sigma_{m2}$), which describe the behavior of a bundle of fibers in a composite reinforced with high-strength fibers under failure conditions. For $\sigma_{m1} = \sigma_{m2}$, the strain-force criterion involves two parameters. The advantages of the two-criteria analysis of quasibrittle fracture are discussed in [11].

2. Sufficient Criterion of Quasibrittle Strength of a Composite with High-Strength Fibers ($\boldsymbol{\sigma_{m1} \ll \sigma_{m2}}$). The internal opening-mode crack in a composite is modeled by a bilateral cut with bonds located in the vicinity of its tip if $\Delta > 0$. We assume that the restriction $K_{\rm I}^0 > 0$ is satisfied for the total SIF (the case of $K_{\rm I}^0 = 0$ is considered in [6]).

The necessary discrete–integral criterion of brittle strength has the following form ($\Delta = 0$):

$$\frac{1}{kr_1} \int\limits_0^{nr_1} \sigma_y(x,0)\,dx \leqslant \sigma_{m1}, \qquad x \geqslant 0. \tag{3}$$

The sufficient discrete–integral criterion of quasibrittle strength has the following form ($\Delta > 0$, $a_{m2} > 0$, and $l = l_0 + \Delta$):

$$\frac{1}{kr_1} \int\limits_0^{nr_1} \sigma_y(x,0)\,dx \leqslant \sigma_{m1}, \qquad x \geqslant 0, \quad 2v^* \leqslant a_{m2}, \quad x \leqslant 0. \tag{4}$$

Here σ_y are the normal stresses on the crack continuation, which have a singular component, Oxy is a Cartesian coordinate system whose origin coincides with the right tip of the fictitious crack, $2l_0$ is the length of the initial crack, $2l$ is the length of the fictitious crack, n and k are numbers ($n \geqslant k$, where k is the number of undamaged fibers), nr_1 is the averaging interval, $(n-k)/n$ is the coefficient that takes into account damaged reinforcement in the averaging interval, $2v = 2v(x,0)$ is the crack-opening displacement, and $2v^*(-\Delta, 0) = a_{m2}$ is the CCO value at which the fiber nearest to the crack center fails.

Outside the pre-fracture zone, the composite material with high-strength fibers is modeled by an orthotropic medium. We use the analytical expressions obtained in [8] for the normal stresses on the continuation of the fictitious crack $\sigma_y(x_1,0)$ and its opening $2v(x_1,0)$:

$$\sigma_y(x_1,0) = \frac{x_1}{\sqrt{x_1^2 - l^2}}\Big(\sigma_\infty - \frac{2\sigma_{m2}}{\pi}\arccos\frac{l_0}{l}\Big) + \sigma_{m2} - \frac{\sigma_{m2}}{\pi}\Big(\arcsin\frac{l^2 - l_0x_1}{l(l_0 - x_1)} + \arcsin\frac{l^2 + l_0x_1}{l(l_0 + x_1)}\Big), \quad |x_1| > l; \tag{5}$$

$$2v(x_1,0) = \frac{4}{E^*}\Big(\sigma_\infty - \frac{2\sigma_{m2}}{\pi}\arccos\frac{l_0}{l}\Big)\sqrt{l^2 - x_1^2} + \frac{2\sigma_{m2}}{\pi E^*}[(x_1 - l_0)F(x_1,l_0) + (x_1 + l_0)F(x_1,-l_0)], \quad |x_1| < l; \tag{6}$$

$$F(x_1,l_0) = \ln\frac{l^2 - l_0x_1 - \sqrt{(l^2 - l_0^2)(l^2 - x_1^2)}}{l^2 - l_0x + \sqrt{(l^2 - l_0^2)(l^2 - x_1^2)}}, \qquad E^* \approx \chi_1E_1 + \chi_2E_2.$$

Here $x_1 = x + l$, i.e., the coordinate origin coincides with the middle points of the initial and fictitious cracks.

An analysis of the solution for stresses (5) shows that the first term in this expression contains a singular component of the stress $\sigma_y(x_1,0)$. The stresses $\sigma_y(x_1,0)$ have a singular component for $\sigma_\infty - (2\sigma_{m2}/\pi)\arccos(l_0/l) > 0$ (below, in particular, this case is considered). If the singular component vanishes [$\sigma_\infty - (2\sigma_{m2}/\pi)\arccos(l_0/l) = 0$], i.e., the Khristianovich hypothesis [12] is valid, the stresses $\sigma_y(x_1,0)$ at the tip of the fictitious crack [at the point $(l,0)$] are finite: $\sigma_y(0,0) = \sigma_{m2}[1 + 2\pi^{-1}\arcsin(l/l_0)]$. We estimate the contribution of various terms to the solution for crack opening (6): for $\sigma_\infty - (2\sigma_{m2}/\pi)\arccos(l_0/l) > 0$, the first term in (6) plays the leading role. We confine ourselves to the simplest asymptotic representations for stresses and crack opening (the right tip of the crack is considered, and all secondary terms are neglected):

$$\sigma_y(x,0) \simeq \sqrt{\frac{l}{2x}}\Big(\sigma_\infty - \frac{2\sigma_{m2}}{\pi}\arccos\frac{l_0}{l}\Big), \qquad x > 0; \tag{7}$$

$$2v(x,0) \simeq \frac{4}{E^*}\Big(\sigma_\infty - \frac{2\sigma_{m2}}{\pi}\arccos\frac{l_0}{l}\Big)\sqrt{2|x|l}, \qquad x < 0. \tag{8}$$

We substitute the asymptotic representation for stresses (7) and crack-opening displacement (8) into the first and second conditions of the sufficient criterion of quasibrittle strength (4). After some obvious transformations of the expressions for the critical parameters σ_∞^*, Δ^*, and $l^* = l_0 + \Delta^*$, we obtain the system of two nonlinear equations

$$\frac{\sqrt{n}}{k}\sqrt{\frac{2l^*}{r_1}}\Big[\frac{\sigma_\infty^*}{\sigma_{m2}} - \frac{2}{\pi}\arccos\Big(1 - \frac{\Delta^*}{l^*}\Big)\Big] \simeq \frac{\sigma_{m1}}{\sigma_{m2}}, \qquad 4\sqrt{2}\,\frac{\sigma_{m2}}{E^*}\Big[\frac{\sigma_\infty^*}{\sigma_{m2}} - \frac{2}{\pi}\arccos\Big(1 - \frac{\Delta^*}{l^*}\Big)\Big]\sqrt{\frac{\Delta^*}{l^*}} \simeq \frac{a_{m2}}{l^*}. \tag{9}$$

System (9) is applicable only to long cracks ($l/r_1 \gg 1$), since the secondary terms in relations (7) and (8) are ignored. We confine ourselves to the quasibrittle approximation where $\Delta^*/l^0 \ll 1$ or $\Delta^*/l \ll 1$ (classification of fracture types by the length of the pre-fracture zone is given in [6]). For quasibrittle fracture, system (9) can be substantially simplified:

$$\frac{\sigma_\infty^*}{\sigma_{m2}} \simeq \frac{k}{\sqrt{n}}\sqrt{\frac{r_1}{2l^*}}\,\frac{\sigma_{m1}}{\sigma_{m2}} + \frac{1}{2\pi}\,\frac{E^*}{\sigma_\infty^*}\,\frac{a_{m2}}{l^*}, \qquad \sqrt{\frac{\Delta^*}{l^*}} \simeq \frac{1}{4\sqrt{2}}\,\frac{E^*}{\sigma_\infty^*}\,\frac{a_{m2}}{l^*}. \tag{10}$$

The critical length of the pre-fracture zone Δ^* is directly related to the limit deformability of high-strength fibers a_{m2}. With an increase in this deformability, the system can sustain higher stresses σ_∞ resulting from partial failure of the matrix. In the limiting case as $a_{m2} \to 0$, we obtain a relation that corresponds to the necessary criterion (3).

3. Discussion. With allowance for relations (9) or (10), we can state that the sufficient discrete–integral criterion (4) is formulated for the problem considered as a three-parameter strain-force criterion for the parameters a_{m2}, σ_{m1}, and σ_{m2}, which describe the strength properties of the material ($\sigma_{m1} \neq \sigma_{m2}$). The other two geometrical parameters r_1 and Δ, which describe the material structure and length of the pre-fracture zone, are involved only in intermediate calculations (the quantity r_1 was taken as a scale in measuring the crack length l_0, l). Equations of the fracture curves for multi-parameter criteria, constructed in Sec. 2 (see also [1, 6]) contain standard strength characteristics of the material. The technique for constructing equations of strength curves with the use of various criteria is the same irrespective of whether the isotropic material exhibits plasticity [6] or partial cracking [1] of the matrix occurs.

Based on experimental data, Ando et al. [13] recommend to use multi-parameter strength criteria in predicting the failure of real ceramic materials: generally, the SIF of cracked solids, when fractured, is not a material constant and can depend on the crack length. As a rule, the failure mode of ceramic materials is brittle or quasibrittle [6]. After the matrix fails, however, the formation of the pre-fracture zone strongly depends on the residual strength of the composite, i.e., the ratio σ_{m2}/σ_{m1} for $\sigma_{m2} \gg \sigma_{m1}$. Even for a small length of the pre-fracture zone Δ for $\sigma_{m2}/\sigma_{m1} > 1$, the critical loads predicted by the necessary (3) and sufficient (4) criteria differ significantly and the dependence of the SIF-type parameter on the crack length becomes pronounced in the sufficient criterion [see (10)].

Serious difficulties arise in describing the failure of solids with V-shaped cuts [14, 15]. According to the classical fracture mechanics, the critical SIF depends on the opening angle of the cut [15]. Probably, it makes sense to ignore some statements of the classical fracture mechanics. It is worth noting that the classical SIF is, undoubtedly, a convenient approximation in the case where singular components of the solutions of classical linear equations of solid mechanics are considered.

4. Principles of Construction of Multi-Parameter Criteria. Based on the Leonov–Panasyuk–Dugdale model [4, 5], we propose the following technique for constructing multi-parameter criteria. We consider a composite with a plastic matrix and reinforcing (possibly, high-strength) fibers. It is assumed that the σ–ε diagram of the composite can be approximated by the relations

$$\sigma = E\varepsilon \text{ for } \varepsilon \leqslant \varepsilon_0, \quad \sigma = \sigma_{m1} \text{ for } \varepsilon_0 \leqslant \varepsilon < \varepsilon_1, \quad \sigma = \sigma_{m2} \text{ for } \varepsilon_1 \leqslant \varepsilon < \varepsilon_2. \tag{11}$$

Here $\sigma_{m1} \neq \sigma_{m2}$ ($\sigma_{m1} = \text{const}$ and $\sigma_{m2} = \text{const}$), and $\varepsilon_1 > \varepsilon_0$. Approximation of the σ–ε diagram depends on Young's modulus E and four independent parameters σ_{m1}, σ_{m2}, $\varepsilon_0 - \varepsilon_1$, and $\varepsilon_0 - \varepsilon_2$, which characterize the strength (σ_{m1} and σ_{m2}) and strain ($\varepsilon_0 - \varepsilon_1$ and $\varepsilon_0 - \varepsilon_2$) properties of the composite material. For the material type considered, the assumptions of the Leonov–Panasyuk–Dugdale model [4, 5] are formulated as follows ($2l_0$ is the initial length of the internal crack): two pre-fracture zones are formed ahead of the crack tips (Δ_1 is the length of the pre-fracture zone of the matrix material and Δ_2 is the length of the pre-fracture zone of the reinforcement matrix material). These zones follow one another and are counted from the fictitious-crack tip: $2l = 2l_0 + 2\Delta$; the overall length of the pre-fracture zone in the vicinity of the fictitious-crack tips is $\Delta = \Delta_1 + \Delta_2$. We assume that the restriction $K_{\rm I} > 0$ is satisfied for the total SIF at the tip of the fictitious crack.

The sufficient discrete-integral criterion can be formulated in the following form ($\Delta = \Delta_1 + \Delta_2 > 0$, $a_{m1} > 0$, $a_{m2} > 0$, and $l = l_0 + \Delta$):

$$\frac{1}{kr_1} \int\limits_0^{nr_1} \sigma_y(x, 0)\, dx \leqslant \sigma_{m1}, \qquad x \geqslant 0,$$
$$2v^* \leqslant a_{m1}, \quad -\Delta_1 \leqslant x \leqslant 0, \qquad 2v^* \leqslant a_{m2}, \quad -\Delta_1 - \Delta_2 \leqslant x \leqslant -\Delta_1. \tag{12}$$

Here $2v^*(-\Delta_1, 0) = a_{m1}$ is the CCO for which the matrix plasticity is exhausted, $2v^*(-\Delta_1 - \Delta_2, 0) = a_{m2}$ is the CCO for which the fiber nearest to the crack center fails, and the remaining notation is the same as in Sec. 2. The crack-opening displacements a_{m1} and a_{m2} are determined by the quantities $\varepsilon_0 - \varepsilon_1$ and $\varepsilon_0 - \varepsilon_2$ from (11) and models that describe the cross-sectional size of the corresponding pre-fracture zones.

The limiting passage from criterion (12) to criterion (4) for $\varepsilon_0 = \varepsilon_1$ ($a_{m1} = 0$) is obvious. Criterion (12) proposed above is a natural generalization of criterion (4). Since the two-parameter criterion has already been constructed for concrete [16], the four-parameter criterion proposed can possibly be used to describe the failure of ferroconcrete in the case where steel reinforcement works beyond the elastic limit.

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